ON HYPERCOMPLEX NUMBER SYSTEMS BELONGING TO AN ARBITRARY DOMAIN OF RATIONALITY*

BY

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In this paper I shall consider only number systems with units $e_1, e_2, \cdots, e_n$, whose constants of multiplication, γ_{ijk}, lie in R, a domain of rationality determined by an arbitrary aggregation of scalars which form a closed system with respect to addition, subtraction, multiplication and division; such systems are said to *belong to* R. The units $e_1, e_2, \cdots, e_n$ will be so chosen that there shall exist between them no linear relation with coefficients in R. In general I shall consider only numbers

$$A = \sum_{i=1}^{n} a_i e_i,$$

of a number system belonging to R, for which the a's lie in R; all such numbers form a closed system with respect to the operations of multiplication, addition and subtraction, and the totality of such numbers is said to constitute the *hypercomplex domain* $\mathfrak{R}(R, e_i)$ in which the numbers A lie. Thus if A and B lie in $\mathfrak{R}(R, e_i)$ so also does $A \pm B$ and also $A \cdot B$, since R contains the constants of multiplication of the system. Moreover, if ρ lies in R, ρA lies in $\mathfrak{R}(R, e_i)$. The introduction of the conception of a domain of rationality necessitates a revision of certain fundamental definitions as usually accepted, such as of reducibility and of equivalence.

A transformation,

$$(e_1', e_2', \cdots, e_n') = T(e_1, e_2, \cdots, e_n),$$

of the units is said to be *rational with respect to* R if the coefficients of T lie in R. The transformation T is always assumed to be linear and with determinant not zero.

Let T be rational with respect to R. Then,

(i) The e''s lie in $\mathfrak{R}(R, e_i)$;

(ii) The hypercomplex domains $\mathfrak{R}(R, e_i)$ and $\mathfrak{R}(R, e_i')$ are identical;

(iii) The system $e_1', e_2', \cdots, e_n'$ belongs to R.

Two number systems are *equivalent with respect to* R if one can be trans-

* Presented to the Society February 25, 1905. Received for publication January 29, 1907.

formed into the other by a transformation rational with respect to R; otherwise *inequivalent with respect to* R.

A number system is *reducible with respect to* R if it is equivalent with respect to R to a system which can be separated into two mutually exclusive subsystems such that the product of each unit of one subsystem by each unit of the other subsystem, both as a prefactor and as a postfactor, is zero; such subsystems are said to be *mutually nilfactorial.*

The general theory of hypercomplex number systems, as usually treated, coincides with the present theory when R contains all real and imaginary scalars, and the theory of real hypercomplex number systems coincides with the present theory when R is restricted to real scalars. The conception which is the basis of the present paper was set forth by H. TABER* in these Transactions, vol. 5 (1904), pp. 509–548. In what follows, unless otherwise explicitly stated, A, A_i, B, etc., will denote hypercomplex numbers lying in $\mathfrak{R}(R, e_i)$ and a, a_i, b, etc., will denote scalars lying in R. Moreover, unless the contrary is stated, any transformation of the units will be assumed to be linear with non-zero determinant and rational with respect to R. In general I shall employ the nomenclature of B. PEIRCE. †

§ 1. *The fundamental equation.*

For any given number

$$A = \sum_{i=1}^{n} a_i e_i$$

there is a smallest positive integer ν for which A, A^2, A^3, $\cdots$, A^ν are linearly related in $\mathfrak{R}(R, e_i)$; this relation

$$\Omega(A) \equiv A^\nu + p_1 A^{\nu-1} + \cdots + p_{\nu-1} A = 0 \qquad (p\text{'s in } R)$$

is called the *fundamental equation of* A. If

$$\phi(A) \equiv A^{\nu'} + p'_1 A^{\nu'-1} + \cdots + p'_{\nu'-1} A = 0 \qquad (p'\text{'s in } R),$$

then ‡ for any scalar λ,

$$\phi(\lambda) \equiv \sum_{p=0}^{\nu'-\nu} c_p \lambda^p \Omega(\lambda) \qquad (c\text{'s in } R).$$

Let

$$\Omega(\lambda) \equiv \lambda^m \cdot [\Omega_1(\lambda)]^{m_1} \cdot [\Omega_2(\lambda)]^{m_2} \cdots [\Omega_s(\lambda)]^{m_s}$$

where $\Omega_1(\lambda)$, $\Omega_2(\lambda)$, $\cdots$, $\Omega_s(\lambda)$ of orders r_1, r_2, $\cdots$, r_s are the distinct factors of

*This paper will be cited under T_1. A second paper by H. TABER, that will soon appear and that I have consulted, will be cited under T_2.

† American Journal of Mathematics, vol. 4 (1882), p. 97. Cf. H. E. HAWKES, these Transactions, vol. 3 (1902), p. 312; and American Journal of Mathematics, vol. 24 (1902), p. 87.

‡ TABER, T_1, p. 51[illegible]

$\Omega(\lambda)$, irreducible with respect to R. The greatest value s may have for any number A in the system will be denoted by $\bar{s}$. Any transformation T, rational in R, of the units, leaves $\bar{s}$ unaltered.* Let m be the greatest value $\mathfrak{m}$ may have for any number A for which $s = \bar{s}$; for $i = 1, 2, \cdots, \bar{s}$, severally, let r_i be the greatest value $\mathfrak{r}_i$ may have for any number A for which $s = \bar{s}$; and for $i = 1, 2, \cdots, \bar{s}$, severally, let m_i be the greatest value $\mathfrak{m}_i$ may have for any number A for which $s = \bar{s}$ and $\mathfrak{r}_i = r_i$.

Among the numbers A for which $s = \bar{s}$ one or more can be found for which simultaneously †

$$\mathfrak{m} = m, \qquad \mathfrak{r}_i = r_i, \qquad \mathfrak{m}_i = m_i \qquad (i=1, 2, \cdots, \bar{s}).$$

I shall assume that the number A has been chosen to fulfill these conditions. Then, corresponding to the irreducible factors $\Omega_i(\lambda)$ of $\Omega(\lambda)$ $(i = 1, 2, \cdots, \bar{s})$, there are $\bar{s}$ independent numbers $I_i = f_i(A)$, lying in $\Re(R, e_i)$, the f_i being polynomials in A with coefficients rational in R, such that

$$I_i I_i = I_i, \qquad I_i I_j = 0 \qquad (i, j=1, 2, \cdots, \bar{s};\ i \neq j).$$

With respect to these $\bar{s}$ idempotent numbers, $I_1, I_2, \cdots, I_{\bar{s}}$, the system may be *regularized*, that is, the units may be so chosen, by a transformation rational in R, that each will fall into one of the following groups of units:

$$\begin{array}{cccccc}
\overline{11}, & \overline{12}, & \cdots, & \overline{1\bar{s}}, & \overline{10}, \\
\overline{21}, & \overline{22}, & \cdots, & \overline{2\bar{s}}, & \overline{20}, \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\overline{\bar{s}1}, & \overline{\bar{s}2}, & \cdots, & \overline{\bar{s}\bar{s}}, & \overline{\bar{s}0}, \\
\overline{01}, & \overline{02}, & \cdots, & \overline{0\bar{s}}, & \overline{00},
\end{array}$$

such that, when any number of the group $\overline{ij}$, lying in $\Re(R, e_i)$, is represented by (ij), $(ij)_1$, $(ij)_2$, etc., the following table gives the group of the product of numbers belonging to the several groups:

	$(ii)_1$	$(ij)_1$	$(ji)_1$	$(jj)_1$
(ii)	$(ii)_2$	$(ij)_2$	0	0
(ij)	0	0	$(ii)_2$	$(ij)_2$
(ji)	$(ji)_2$	$(jj)_2$	0	0
(jj)	0	0	$(ji)_2$	$(jj)_2$

Moreover, for $1 \leqq i \leqq \bar{s}$:

* TABER, T_2.

† TABER, T_2.

(1) The idempotent number I_i belongs to the groups $\overline{ii}$ the units of which constitute a subsystem; I_i is the modulus of this subsystem, and is the only idempotent number in it lying in $\Re(R, e_i)$.

(2) The nilpotent numbers of the group $\overline{ii}$ lying in $\Re(R, e_i)$, constitute an invariant subsystem of the system $\overline{ii}$, that is, the product of any such number and any number of the group $\overline{ii}$, lying in $\Re(R, e_i)$, both as prefactor and as postfactor, is a nilpotent number of the group $\overline{ii}$, lying in $\Re(R, e_i)$.

(3) Among the numbers of the group $\overline{ii}$ are three numbers, A_i, $\bar{A}_i$ and A'_i, polynomials in A with coefficients rational in R, whose fundamental equations are, respectively,

$$\omega_i(A_i) = 0, \qquad \bar{\omega}_i(\bar{A}_i) = 0, \qquad \omega'_i(A'_i) = 0,$$

where

$$\omega_i(\lambda) \equiv \lambda[\Omega_i(\lambda)]^{m_i}, \qquad \bar{\omega}_i(\lambda) \equiv \lambda[\Omega_i(\lambda)], \qquad \omega'_i(\lambda) \equiv \lambda^{m_i}.$$

The numbers I_i, $\bar{A}_i$, $\bar{A}_i^2$, $\cdots$, $\bar{A}_i^{r_i-1}$ are independent and

$$B \equiv c_0 I_i + c_1 \bar{A}_i + c_2 \bar{A}_i^2 + \cdots + c_{r_i-1} \bar{A}_i^{r_i-1}$$

has a reciprocal with respect to I_i unless each $c_i = 0$.

(4) $I_i(ij) = (ij)$, $(ji)I_i = (ji)$ $(j = 0, 1, 2, \cdots, \bar{s})$.

(5) The units of $\overline{00}$ form a nilpotent system, in which there is a number A_0, a polynomial in A with coefficients rational in R, such that its fundamental equation is

$$\omega_0(A_0) = 0,$$

where

$$\omega_0(\lambda) \equiv \lambda^m.$$

Finally we may substitute for A the number $A_0 + A_1 + A_2 + \cdots + A_{\bar{s}}$, for which $\mathfrak{m} = m$, $\mathfrak{r}_i = r_i$, $\mathfrak{m}_i = m_i$ $(i = 1, 2, \cdots, \bar{s})$. Then, for $i = 0, 1, 2, \cdots, \bar{s}$, A_i is a polynomial in this new number $\sum A_i$ (which from this point on we shall call A), with coefficients rational in R.*

Within $\Re(R, e_i)$ a nilpotent system belonging to R is of *order* p if the pth power of every number of the system is zero, while the $(p-1)$th power of some number of the system is not zero.

THEOREM I. *For* $1 \leqq i \leqq \bar{s}$, *the nilpotent numbers of the system* $\overline{ii}$ *commutative with* $\bar{A}_i$ *form a nilpotent subsystem whose order is* m_i. *The units of the system* $\overline{00}$ *form a nilpotent subsystem whose order is* m.

The nilpotent numbers of the group $\overline{ii}$ constitute a subsystem, and therefore the nilpotent numbers of $\overline{ii}$ commutative with $\bar{A}_i$ form a nilpotent subsystem. Suppose there were in $\overline{ii}$ a number N_i such that

$$N_i\bar{A}_i = \bar{A}_i N_i, \qquad N_i^{m_i+q} \neq 0, \qquad N_i^{m_i+q+1} = 0.$$

* For the proof of these theorems see TABER, T_2.

Now the numbers

$$\bar{A}_i, \bar{A}_i^2, \cdots, \bar{A}_i^{r_i-1}; \qquad \bar{A}_i N_i, \bar{A}_i^2 N_i, \cdots, \bar{A}_i^{r_i-1} N_i; \qquad \cdots; \qquad \cdots;$$
$$\bar{A}_i N_i^{m_i+q}, \bar{A}_i^2 N_i^{m_i+q}, \cdots, \bar{A}_i^{r_i-1} N_i^{m_i+q}$$

are independent. For, otherwise, on multiplying by a proper power of N_i, we would get a relation of the form

$$(c_1 \bar{A}_i + c_2 \bar{A}_i^2 + \cdots + c_{r_i-1} \bar{A}_i^{r_i-1}) N_i^{m_i+q} = 0$$

where the c's are not all zero; but in that event $\sum_{j=1}^{r_i-1} c_j \bar{A}_i^j$ has a reciprocal with respect to I_i [see § 1 (1)], and therefore $N_i^{m_i+q} = 0$, which is contrary to hypothesis. Let

$$B_i = \bar{A}_i + N_i,$$

and let

$$\phi(B_i) \equiv B_i^\mu + b_2 B_i^{\mu-1} + \cdots + b_\mu B_i = 0$$

be the fundamental equation of B_i. Since $\bar{A}_i$ and N_i are commutative

$$\phi(B_i) = \phi(\bar{A}_i + N_i) = \phi(\bar{A}_i) + \phi'(\bar{A}_i) \cdot N_i + \frac{\phi''(\bar{A}_i) \cdot N_i^2}{2!}$$
$$+ \cdots + \frac{\phi^{(m_i+q)}(\bar{A}_i) \cdot N_i^{m_i+q}}{(m_i + q)!} = 0,$$

where $\phi^p(\lambda) \equiv d^p \phi(\lambda)/d\lambda^p$. Therefore

$$\phi(\bar{A}_i) = \phi'(\bar{A}_i) = \cdots = \phi^{(m_i+q)}(\bar{A}_i) = 0;$$

for otherwise there would be a relation among the $\bar{A}_i^k N_i^j$'s. Whence it follows (see § 1, first part) that

$$\phi(\lambda) \equiv \lambda [\Omega_i(\lambda)]^{m_i+q+1}.$$

Let

$$B = A_0 + A_1 + \cdots + A_{i-1} + B_i + A_{i+1} + \cdots + A_{\bar{s}}.$$

Then, since A_j $(j = 0, 1, 2, \cdots, \bar{s})$ is a number of the group $\bar{\bar{j}}$, we have

$$B^p = A_0^p + A_1^p + \cdots + A_{i-1}^p + B_i^p + A_{i+1}^p + \cdots + A_{\bar{s}}^p,$$

and if $\psi(B) = 0$ is the fundamental equation of B,

$$\psi(A_0) = \psi(A_1) = \cdots = \psi(A_{i-1}) = \psi(B_i) = \psi(A_{i+1}) = \cdots = \psi(A_{\bar{s}}) = 0.$$

Therefore

$$\psi(\lambda) \equiv \lambda^m [\Omega_1(\lambda)]^{m_1} \cdots [\Omega_{i-1}(\lambda)]^{m_{i-1}} \cdot [\Omega_i(\lambda)]^{m_i+q+1} [\Omega_{i+1}(\lambda)]^{m_{i+1}} \cdots [\Omega_{\bar{s}}(\lambda)]^{m_{\bar{s}}},$$

and thus there is a number B for which

$$s = \bar{s}, \ \mathfrak{r}_j = r_j \ (j = 1, 2, \cdots, \bar{s}), \ \mathfrak{m}_k = m_k \ (k = 1, 2, \cdots, i-1, i+1, \cdots, \bar{s}),$$

and $\mathfrak{m}_i = m_i + q + 1 > m_i$, which is contrary to hypothesis. Consequently no such number as N_i exists in $\bar{\bar{i}}$.

But by (3) there is a number A'_i, in $\bar{i}\bar{i}$, such that $A_i'^{m_i-1} \neq 0$, $A_i'^{m_i} = 0$, and such that $\bar{A}_i$ and A'_i are commutative, both being polynomials in A. Hence follows the first part of the theorem.

Suppose there were a number N_0 in $\overline{00}$ such that

$$N_0^{m+q} \neq 0, \qquad N_0^{m+q+1} = 0.$$

Let $B = N_0 + A_1 + A_2 + \cdots + A_{\bar{s}}$, and let its fundamental equation be $\phi(B) = 0$. Then, since $N_0, A_1, A_2, \cdots, A_{\bar{s}}$ are mutually nilfactorial,

$$\phi(B) = \phi(N_0) + \phi(A_1) + \phi(A_2) + \cdots + \phi(A_{\bar{s}}) = 0.$$

Hence follows

$$\phi(N_0) = \phi(A_1) = \phi(A_2) = \cdots = \phi(A_{\bar{s}}) = 0,$$

and therefore

$$\phi(\lambda) \equiv \lambda^{m+q+1}[\Omega_1(\lambda)]^{m_1} \cdot [\Omega_2(\lambda)]^{m_2} \cdot \cdots \cdot [\Omega_{\bar{s}}(\lambda)]^{m_{\bar{s}}}.$$

and thus there is a number, viz., B, for which

$$s = \bar{s}, \qquad \mathfrak{r}_i = r_i, \qquad \mathfrak{m}_i = m_i \ (i = 1, 2, \cdots, \bar{s}), \qquad \mathfrak{m} = m + q + 1 > m,$$

which is contrary to hypothesis. Hence no such number as N_0 exists in $\overline{00}$.

There is, however, by (5), a number A_0 in $\overline{00}$ such that

$$A_0^{m-1} \neq 0, \qquad A_0^m = 0.$$

Hence follows the second part of the theorem.

§ 2. *Reducibility.*

THEOREM II. *The number of units in the group* $\bar{i}\bar{i}$ $(i = 1, 2, \cdots \bar{s},)$ *is a multiple of* r_i, *and is not less than* $r_i m_i$.

Let B_i be any number in $\bar{i}\bar{i}$. Then

$$B_i, \bar{A}_i B_i, \bar{A}_i^2 B_i, \cdots, \bar{A}_i^{r_i-1} B_i$$

are independent numbers in $\bar{i}\bar{i}$ and may be used as r_i of the units. For, if

$$\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p B_i = 0,$$

where the c's are not all zero, we get, on multiplying by the reciprocal, with respect to I_i, of $\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p$,

$$B_i = 0,$$

which is contrary to hypothesis. If there is a number B'_{ii}, in $\bar{i}\bar{i}$, independent of

$$B_i, \bar{A}_i B_i, \bar{A}_i^2 B_i, \cdots, \bar{A}_i^{r_i-1} B_i$$

then

$$B_i, \bar{A}_i B_i, \bar{A}_i^2 B_i, \cdots, \bar{A}^{r_i-1} B_i;\ B'_i, \bar{A}_i B'_1, \bar{A}_i^2 B'_i, \cdots, \bar{A}_i^{r_i-1} B'_i$$

are independent numbers in $\bar{i}\bar{i}$ and may be used as $2r_i$ of the units. For, if

$$\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p B_i + \sum_{p=0}^{r_i-1} c'_p \bar{A}_i^p B'_i = 0,$$

where the c''s are not all zero, we get on multiplying by the reciprocal of $\sum_{p=0}^{r_i-1} c'_p \bar{A}_i^p$, with respect to I_i, a linear relation between B'_i and B_i, $\bar{A}_i B_i, \cdots,$ $\bar{A}_i^{r_i-1} B_i$, which is contrary to hypothesis. If there is a number B''_i, in $\overline{ii}$, independent of the $2r_i$ units now chosen, we proceed as before. By continuing this process we can show that the number n_i of units in $\overline{ii}$ is a multiple of r_i.

The numbers

$$\begin{array}{lllll} I_i, & \bar{A}_i, & \bar{A}_i^2, & \cdots, & \bar{A}_i^{r_i-1}; \\ A'_i, & \bar{A}_i A'_i, & \bar{A}_i^2 A'_i, & \cdots, & \bar{A}_i^{r_i-1} A'_i; \\ A_i'^2, & \bar{A}_i A_i'^2, & \bar{A}_i^2 A_i'^2, & \cdots, & \bar{A}_i^{r_i-1} A_i'^2; \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ A_i'^{m_i-1}, & \bar{A}_i A_i'^{m_i-1}, & \bar{A}_i^2 A_i'^{m_i-1}, & \cdots, & \bar{A}_i^{r_i-1} A_i'^{m_i-1}; \end{array}$$

are independent and may be used as $m_i r_i$ units in $\overline{ii}$. For otherwise, on multiplying by a proper power of A'_i we should get a relation of the form

$$(c_0 I_i + c_1 \bar{A}_i + c_2 \bar{A}_i^2 + \cdots + c_{r_i-1} \bar{A}_i^{r_i-1}) A_i'^{m_i-1} = 0,$$

where the c's are not all zero; but in this case $\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p$ has a reciprocal with respect to I_i and therefore

$$A_i'^{m_i-1} = 0,$$

which is contrary to hypothesis, whence it follows that $n_i \geqq m_i r_i$.

THEOREM III. *The number of units in the group $\overline{00}$ is not less than $m-1$.*

Indeed, the independent numbers

$$A_0,\ A_0^2,\ \cdots,\ A_0^{m-1}$$

in $\overline{00}$ may be chosen as $m-1$ of the units.

THEOREM IV. *The number, n_{ij}, of units in the group $\overline{ij}$ $(i, j = 1, 2, \cdots, \bar{s})$ is a multiple both of r_i and of r_j; and thus $n_{ij} \geqq r_i$, $n_{ij} \geqq r_j$ if $n_{ij} \neq 0$. Moreover n_{i0} and n_{0i} $(i = 1, 2, \cdots, \bar{s})$ are multiples of r_i.*

If there is one number B_{ij} in $\overline{ij}$ $(i = 1, 2, \cdots, \bar{s};\ j = 0, 1, 2, \cdots, \bar{s})$, then

$$B_{ij},\ \bar{A}_i B_{ij},\ \bar{A}_i^2 B_{ij},\ \cdots,\ \bar{A}_i^{r_i-1} B_{ij}$$

are r_i independent numbers in $\overline{ij}$. For if

$$\sum_{p=1}^{r_i-1} c_p \bar{A}_i^p B_{ij} = 0,$$

where the c's are not all zero, we get on multiplying by the reciprocal of

$$\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p$$

with respect to I_i,

$$B_{ij} = 0,$$

which is contrary to hypothesis. If B'_{ij} in $\overline{ij}$ is independent of

$$B_{ij},\ \bar{A}_i B_{ij},\ \bar{A}_i^2 B_{ij},\ \cdots,\ \bar{A}_i^{r_i-1} B_{ij},$$

we get the following $2r_i$ independent numbers in $\overline{ij}$.

$$B_{ij},\ \bar{A}_i B_{ij},\ \bar{A}_i^2 B_{ij},\ \cdots,\ \bar{A}_i^{r_i-1} B_{ij};\ B'_{ij},\ \bar{A}_i B'_{ij},\ \bar{A}_i^2 B'_{ij}\ \cdots,\ \bar{A}_i^{r_i-1} B'_{ij}.$$

Proceeding as in the proof of Theorem II, we may show that n_{ij} is a multiple of r_i. Likewise it can be shown that n_{ij} is a multiple of r_j ($i = 0, 1, 2, \cdots, \bar{s}$; $j = 1, 2, \cdots, \bar{s}$).

THEOREM V. *If the number of units in* $\overline{i0}$ (*or* $\overline{0i}$) *is just* r_i ($i = 1, 2, \cdots, \bar{s}$), *then the product of any number in* $\overline{i0}$ *into any number in* $\overline{00}$ (*or any number in* $\overline{00}$ *into any number in* $\overline{0i}$) *is zero.*

Let $(i0)_1$ be any number in $\overline{i0}$. Then, as in the proof of Theorem IV, the numbers

$$(i0)_1,\ \bar{A}_i(i0)_1,\ \bar{A}_i^2(i0)_1,\ \cdots,\ \bar{A}_i^{r_i-1}(i0)_1$$

are independent and may be chosen as the r_i units. Then for any number (00) in $\overline{00}$,

$$(i0)_1(00) = [c_0 I_i + c_1 \bar{A}_i + c_2 \bar{A}_i^2 + \cdots + c_{r_i-1} A_i^{r_i-1}]\,(i0)_1$$

since this product must also lie in $\overline{i0}$. From this follows

$$\left[\sum_{p=0}^{r_i-1} c_p \bar{A}_i^p\right]^m \cdot (i0)_1 = (i0)_1 \cdot (00)^m = 0.$$

Therefore, since $(i0)_1 \neq 0$, each $c_p = 0$. Thus

$$(i0)_1 \cdot (00) = 0,$$

whence follows the theorem.

THEOREM VI. *If there are units neither in* $\overline{ij}$ *nor in* $\overline{ji}$, *for* $i = \alpha_1, \alpha_2, \cdots, \alpha_p$ *and* $j = \beta_1, \beta_2, \cdots, \beta_{p'}$, *where* $p + p' = \bar{s}$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_{p'}$ *are all distinct, then the number system is reducible.*

For, under these conditions, all the units in the groups $\overline{ij}$ ($i, j = \alpha_1, \alpha_2, \cdots, \alpha_p$), constitute a subsystem, and all the remaining units, viz., all those in the groups $\overline{ij}$ ($i, j = \beta_1, \beta_2, \cdots, \beta_{p'}$), constitute a second subsystem and these two subsystems are mutually nilfactorial.

THEOREM VII. *If* $m = 1$ *and*

$$n < \sum_{i=1}^{\bar{s}} r_i(m_i + 1) - r_j \qquad (j = 1, 2, \cdots, s),$$

the system is reducible.

Assuming that $m = 1$ and that the system is irreducible, we will find a lower limit to n. By Theorem I there is no unit in $\overline{00}$. There are units in at least one of the groups $\overline{\bar{s}\alpha}$ or $\overline{\alpha\bar{s}}$ for a positive integer $\alpha < \bar{s}$; for, if not, the system is reducible by Theorem VI, which is contrary to hypothesis. Let those groups

$\overline{\bar{s}\alpha_1}$ or $\overline{\alpha_1\bar{s}}$ $(\alpha_1 \neq \bar{s})$ that contain one or more units be given by

$$\alpha_1 = a_{11},\, a_{12},\, a_{13},\, \cdots,\, a_{1p_1}.$$

If $p_1 < \bar{s} - 1$, then for some integer a_2 among $a_{11}, a_{12}, \cdots, a_{1p_1}$ there are units in $\overline{a_2\alpha_2}$ or $\overline{\alpha_2 a_2}$ for at least one integer $\alpha_2 (\alpha_2 < \bar{s})$ other than $a_{11}, a_{12}, \cdots, a_{1p_1}$. For if not, by Theorem VI the given system is reducible, which is contrary to hypothesis. Let the values of α_2 giving those groups $\overline{a_2\alpha_2}$ or $\overline{\alpha_2 a_2}$ that contain one or more units be given by

$$\alpha_2 = a_{21},\, a_{22},\, \cdots,\, a_{2p_2},$$

where none of these integers are included among $\bar{s}, a_{11}, a_{12}, \cdots, a_{1p_1}$. We have $p_1 \geqq 1$, and we have shown that if $p_1 < \bar{s} - 1$, then $2 \leqq p_1 + p_2 \leqq \bar{s} - 1$. If $p_1 + p_2 < \bar{s} - 1$, then for some integer a_3 among $a_{11}, a_{12}, \cdots, a_{1p_1}$; $a_{21}, a_{22}, \cdots, a_{2p_2}$, there are units in $\overline{a_3\alpha_3}$ or $\overline{\alpha_3 a_3}$ for some integer $\alpha_3 < \bar{s}$ other than $a_{11}, a_{12}, \cdots, a_{1p_1}$; $a_{21}, a_{22}, \cdots, a_{2p_2}$. For otherwise, the system is reducible by Theorem VI; and $3 \leqq p_1 + p_2 + p_3 \leqq \bar{s} - 1$, where $\alpha_3 = a_{31}, a_{32}, \cdots a_{3p_3}$, none of these integers appearing among $a_{11}, a_{12}, \cdots, a_{1p_1}$; $a_{21}, a_{22}, \cdots, a_{2p_2}$. Repeating this process t times for some integer $t \geqq 1$, we must have

$$p_1 + p_2 + p_3 + \cdots + p_t = \bar{s} - 1.$$

Thus each positive integer less than $\bar{s}$ and greater than 0 appears once and but once among the integers a_{11}, a_{12}, a_{1p_1}; $a_{21}, \cdots, a_{2p_2}$; $\cdots$; $\cdots$; $a_{t1}, a_{t2}, \cdots, a_{tp_t}$; provided 0 does not appear among these numbers (if 0 does appear t must be chosen so that $p_1 + p_2 + \cdots + p_t = \bar{s}$ in which case all the integers $0, 1, 2, \cdots, \bar{s} - 1$ will appear once and but once); corresponding to each of these integers there is one group $\overline{\beta_i i}$, or $\overline{i\beta_i}$, or there are two groups $\overline{\beta_i i}$ and $\overline{i\beta_i}$ $(i = 1, 2, \cdots, \bar{s} - 1;\ \beta_i \neq i)$, which contain one or more units. Then, by Theorem III, the total number of units in these groups is not less than $\sum_{i=1}^{\bar{s}-1} r_i$.

By Theorem II, the total number of units in the groups $\overline{ii}$ $(i = 1, 2, \cdots, \bar{s})$ is not less than $\sum_{i=1}^{\bar{s}} r_i m_i$, whence it follows that n cannot be less than $\sum_{i=1}^{\bar{s}} r_i m_i + \sum_{i=1}^{\bar{s}-1} r_i$. That is, if

$$n < \sum_{i=1}^{\bar{s}} r_i(m_i + 1) - r_j \qquad (j = 1, 2, \cdots, \bar{s}),$$

the system is reducible.

THEOREM VIII. *If* $m > 1$, *and*

$$n < \sum_{i=1}^{\bar{s}} r_i(m_i + 1) + r_{\bar{\imath}} + m - 1,$$

where the subscripts have been so chosen that $r_{\bar{\imath}} \leqq r_k$ $(k = 1, 2, \cdots, \bar{s})$, *the system is reducible.*

Assuming the system to be irreducible, and repeating the process in the demonstration of Theorem VII, we find that units exist in a group $\overline{\beta_i i}$ or $\overline{i\beta_i}$, or in two groups $\overline{\beta_i i}$ and $\overline{i\beta_i}$ for i successively equal to $0, 1, 2, \cdots, \bar{s}-1$ $(\beta_i \neq i)$. Denote this set of groups by Γ. For $i \neq 0$ there are in the group $\overline{\beta_i i}$ (or $\overline{i\beta_i}$), at least as many as r_i units; and for $i = 0$ there are at least as many as r_{β_0} units in the group.

Now if there are more than $\bar{s}$ groups in Γ, there must be two groups for at least one value of i, as $i = \alpha$. And therefore in this case the total number of units in Γ is not less than

$$r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_\alpha \, (\alpha \neq 0), \qquad r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_{\beta_0} \, (\alpha = 0).$$

If there are just $\bar{s}$ groups in Γ and units in one or more groups $\overline{kl}$ $(k \neq l)$, not in Γ, then the total number of units in all the groups $\overline{ij}$ $(i \neq j)$ is not less than

$$r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_k (k \neq 0), \qquad r_{\beta_0} + \sum_{i=1}^{\bar{s}-1} r_i + r_l \, (k = 0).$$

If there are just $\bar{s}$ groups in Γ and units in no group $\overline{kl}$ $(k \neq l)$, not in Γ, the only existing group $\overline{0p}$ (or $\overline{p0}$) is the single group $\overline{0\beta_0}$ (or $\overline{\beta_0 0}$) in Γ; and therefore if each number in it is nilfactorial to every number $\overline{00}$, the system is reducible. But this holds unless the group $\overline{0\beta_0}$ (or $\overline{\beta_0 0}$) contains $2r_{\beta_0}$ units. For, otherwise, by Theorem V, we have

$$(00)\cdot(0\beta_0) = 0 \; [\text{or } (\beta_0 0)\cdot(00) = 0].$$

Therefore in this case the total number of units in Γ is not less than

$$\sum_{i=1}^{\bar{s}-1} r_i + 2r_{\beta_0}.$$

Hence it follows, since $r_{\bar{s}} \leqq r_k$ $(k = 1, 2, \cdots, \bar{s})$, that in any case the total number of units in the groups $\overline{ij}$ $(i, j = 0, 1, 2, \cdots, \bar{s};\ i \neq j)$ is not less than

$$\sum_{i=1}^{\bar{s}} r_i + r_{\bar{s}}.$$

By Theorem II the total number of units in the groups $\overline{ii}$ $(i = 1, 2, \cdots, \bar{s})$ is not less than $\sum_{i=1}^{\bar{s}} r_i m_i$. By Theorem III the number of units in the group $\overline{00}$ is not less than $m - 1$. Therefore, in any irreducible number system where $m > 1$, $r_{\bar{s}} \leqq r_i$ $(i = 1, 2, \cdots, \bar{s})$,

$$n \leqq \sum_{i=1}^{\bar{s}} r_i (m_i + 1) + r_{\bar{s}} + m - 1.$$

Hence the theorem is proved.

I shall next enumerate the number systems belonging to an arbitrary domain R to which, or to the reciprocals of which, all irreducible number systems in less than five units, and belonging to R, may be transformed by linear transformations rational in R.

§ 3. *Nilpotent systems*, $n \leqq 4$.

If there is no idempotent number in a system it is nilpotent and all numbers in it are nilpotent.*

The multiplication tables of all the nilpotent number systems in n units $e_1, e_2, \cdots, e_n$, to which, or to the reciprocals of which, all nilpotent systems in n-units can be transformed by transformations rational in R, can be found from those systems of the same kind in $n-1$ units $e_1, e_2, \cdots, e_{n-1}$, by adding to each product $e_i e_j = \sum_{k=1}^{n-1} \gamma_{ijk} e_k$ of units in the multiplication table of the system $e_1, e_2, \cdots, e_{n-1}$, a term consisting of $\gamma_{ij} e_n$, where γ_{ij} is a parameter that remains to be determined; the product of any number of the system $e_1, e_2, \cdots, e_n$ and e_n used either as a prefactor or a postfactor being written zero.†

Some of these parameters can be fixed by using the associative law for multiplication, and by simple transformations in which the new parameters are always found by solving linear equations (thus, no irrationality is introduced by these new parameters). For $n \leqq 4$, all the irreducible nilpotent number systems belonging to any domain can be shown equivalent to, or equivalent to a reciprocal of one or another of the systems whose multiplication tables are given below. When the domain R is more precisely defined the number of these forms and the number of parameters in them may be reduced still further. All products not written are zero.

$n = 1$. $e_1^2 = 0$.

$n = 2$. $e_1^2 = e_2$.

$n = 3$. $e_1^2 = e_3$, $e_2 e_1 = e_3$, $e_2^2 = ae_3$; $e_1^2 = e_3$, $e_2^2 = ae_3$; $e_1 e_2 = -e_2 e_1 = e_3$; $e_1^2 = e_2$, $e_1 e_2 = e_2 e_1 = e_3$.

$n = 4$. $e_1^2 = ae_4$, $e_2 e_3 = be_4 = e_3^2$, $e_3 e_1 = e_4 = -e_3 e_2$; $ae_1^2 = e_2 e_1 = e_2^2 = e_3 e_2 = be_3^2 = e_4$; $e_1^2 = e_3 e_2 = e_4$; $ae_1^2 = e_2 e_1 = e_2^2 = be_3^2 = e_4$; $ae_1^2 = e_2^2 = be_3^2 = e_4$; $e_1^2 = e_3$, $e_2 e_1 = e_4$, $e_1 e_2 = e_3 + ae_4$, $e_2^2 = be_3 + ce_4$; $e_1^2 = e_3$, $e_2 e_1 = e_4$, $e_2^2 = ae_3 + e_4$; $e_1^2 = e_3$, $ae_2^2 = e_2 e_1 = e_4$; $e_1^2 = e_3$, $e_2^2 = e_4$; $e_1^2 = e_3$, $e_1 e_2 = ae_2 e_1 = e_2^2 = e_4$; $e_1^2 = e_3$, $e_1 e_2 = ae_2 e_1 = e_4$; $e_1^2 = ae_2^2 = e_3$,

* In accordance with what has been said above we refer here to nilpotent systems belonging to R, and to numbers lying in $\mathfrak{N}(R, e_i)$; for proof of this theorem see TABER, T_1, pp. 512, 525.

† Cf. H. E. HAWKES, Mathematische Annalen, vol. 58, p. 369.

In passing I note an error that appears in his paper. His Theorem VI, p. 369, states that "If two systems are deleted by the same method, $\cdots$ and the deleted systems are inequivalent, the original systems are inequivalent." But on deleting by his method the unit e_4 from each of the two equivalent systems,

$$e_1^2 = e_3,\ e_1 e_2 = e_4,\ e_2 e_1 = ae_4,\ e_2^2 = e_4 \quad \text{and} \quad e_1^2 = e_4,\ e_1 e_2 = e_3,\ e_2 e_1 = ae_3,\ e_2^2 = e_3,$$

we get the two inequivalent systems, $e_1^2 = e_3$ and $e_1 e_2 = e_3$, $e_2 e_1 = ae_3$, $e_2^2 = e_3$. (In these systems the products not written are zero.) The error in the proof given for this theorem lies in the failure to show that the determinant of the equations of transformation which, as HAWKES says, must exist between the units of the deleted systems, is not identically zero. That this can not be shown in general follows from the example just given.

$$e_2e_1 = ae_1e_2 = e_4;\ e_1^2 = e_3,\ e_2^2 = ae_3 + e_4,\ e_1e_2 = ae_4,\ e_2e_1 = e_4;$$
$$e_1^2 = e_3,\ e_1e_3 = ae_2e_1 = e_2^2 = e_3^2 = e_4;\ e_1^2 = e_3,\ e_1e_3 = e_4 = e_2e_1;$$
$$e_i = e_1^i\ (i = 1, 2, 3, 4).$$

§ 4. *Systems composed of a modulus and a nilpotent system.*

All systems in n units or less, composed of a modulus and a nilpotent system, can be found either among the systems obtained by annexing a modulus to the nilpotent systems in less than n units, or among the systems which are obtained by taking the units of two or more nilpotent systems, writing the products of any two units in different systems as zero and annexing a modulus to the nilpotent system thus formed. It is necessary to bring in these later systems since in the enumeration of nilpotent systems reducible systems were discarded and since the reducible nilpotent system produces, on having a modulus annexed, an irreducible system.

§ 5. *Enumeration of all number systems, in less than five units, belonging to an arbitrary domain R.*

Let A be chosen as in § 1. If $\Omega(A) = 0$ is the fundamental equation of A, let us examine, for $n \leqq 4$, the possible forms of

$$\Omega(\lambda) \equiv \lambda^m [\Omega_1(\lambda)]^{m_1} [\Omega_2(\lambda)]^{m_2} \cdot \cdots \cdot [\Omega(\lambda)]^{m_{\bar{s}}},$$

$\Omega_i(\lambda)$ being of order r_i in λ. By means of Theorems VII and VIII we find the possible combinations of values of $\bar{s}$, m, m_i, r_i $(i = 1, 2, \cdots, \bar{s})$ for irreducible number systems in n units; these are given in the following table:

n	m	$\bar{s}$	r_1	m_1	r_2	m_2
1	2	0				
1	1	1	1	1		
2	3	0				
2	2	0				
2	1	1	2	1		
2	1	1	1	2		
2	1	1	1	1		
3	4	0				
3	3	0				
3	2	0				
3	1	2	1	1	1	1
3	1	1	3	1		
3	1	1	1	3		
3	1	1	1	2		
3	1	1	1	1		

n	m	$\bar{s}$	r_1	m_1	r_2	m_2
4	5	0				
4	4	0				
4	3	0				
4	2	1	1	1		
4	2	0				
4	1	2	1	2	1	1
4	1	2	1	1	1	1
4	1	1	4	1		
4	1	1	2	2		
4	1	1	2	1		
4	1	1	1	4		
4	1	1	1	3		
4	1	1	1	2		
4	1	1	1	1		

To distinguish the various cases let $[abc_1^{d_1}c_2^{d_2}\cdots c_{\bar{s}}^{d_{\bar{s}}};k]$ designate the kth number system among those for which $n=a, m=b, r_i=c_i, m_i=d_i$ $(i=1,2,\cdots,\bar{s})$. All products omitted are zero.

The following systems are nilpotent and are found in § 3:

$$[12;1];\ [23;1],\ [22;k];\ [34;1],\ [33;k],\ [32,k];\ [45;1],\ [44;k],\ [43;k],\ [42;k].$$

The following systems made up of a modulus and a nilpotent system are found by means of § 4:

$$[211^2;1];\ [311^3;1],\ [311^2;k]\ (k>1),\ [311^1;k]\ (k>3);\ [411^4;1],\ [411^3;k]\ (k>1),\ [411^2;k]\ (k>5).$$

$[111^1;1]$. $e_1^2=e_1$.

$[212^1;1]$. $e_1e_i=e_ie_1=e_i$ $(1=1,2)$, $e_2^2=pe_1$ ($p=$ not-square in R).

When R is the domain of real scalars we may put $p=-1$, getting the ordinary complex number system considered as belonging to the domain of real scalars.

$[211^1;1]$. $e_1e_i=e_i$ $(i=1,2)$.

$[311^11^1;1]$. $e_1e_i=e_i$ $(i=1,2)$, $e_ie_3=e_i$ $(i=2,3)$.

$[313^1;1]$. $e_ie_1=e_1e_i=e_i=e_2^{i-1}$ $(i=1,2,3)$, $e_2^3-p_1e_2-p_2I=0$, where $\lambda^3-p_1\lambda-p_2=0$ is irreducible in R.

$[311^2;1]$. $e_1e_i=e_i$ $(i=1,2,3)$, $e_2e_1=e_2$.

$[311^1;1]$. $e_1e_i=e_i$ $(i=1,2,3)$.

$[311^1;2]$. $e_1e_i=e_i$ $(i=1,2)$, $e_3e_1=e_3$.

$[421^1;1]$. $e_1e_i=e_i$, $e_3e_i=e_{i+2}$ $(i=1,2)$.

Here we have chosen $e_1=I_1$, $e_2=(10)$, $e_3=(01)$, $e_4=(00)$; by means of Theorems I and V, the products can easily be shown to be as written.

$[421^1;2]$. $e_1e_i=e_i$ $(i=1,2,3)$ $e_2e_4=e_3$.

Here we have chosen $e_1=I_1$, $e_2=(10)$, $e_3=(10)\cdot(00)$, $e_4=(00)$. If $e_3=ae_2=e_2e_4$, then $0=e_3e_4=ae_2e_4=a^2e_2$; and therefore $a=0$, which makes the system reducible. That is, e_2 and e_3 may be properly chosen as independent units if the system is irreducible.

$[411^21^1;1]$. $e_1e_i=e_i$ $(i=1,2,3)$, $e_ie_4=e_i$ $(i=3,4)$.

$[411^11^1;1]$. $e_1e_i=e_i$, $e_2e_{i+2}=e_i$, $e_3e_i=e_{i+2}$, $e_4e_{i+2}=e_{i+2}$ $(i=1,2)$.

$[411^11^1;2]$. $e_1e_i=e_i$, $e_4e_{i+2}=e_{i+2}$ $(i=1,2)$, $e_2e_4=e_2$, $e_3e_1=e_3$.

$[411^11^1;3]$. $e_1e_i=e_i$ $(i=1,2,3)$, $e_ie_4=e_i$ $(i=2,3,4)$.

$[414^1;1]$. $e_ie_1=e_1e_i=e_i=e_2^{i-1}$ $(i=1,2,3,4)$, $e_2^4-p_1e_2^2-p_2e_2-p_3I=0$, $\lambda^4-p_1\lambda^2-p_2\lambda-p_3=0$ being irreducible in R.

$[412^2;1]$. $e_ie_1=e_1e_i=e_i$ $(i=1,2,3,4)$, $e_2^2=pe_1$, $\lambda^2-p=0$ being irreducible in R, $e_2e_3=e_3e_2=e_4$, $e_2e_4=e_4e_2=pe_3$.

In the systems $[412^1;\ k]$ we have

$$\Omega(\lambda) \equiv \lambda(\lambda^2 - p) \qquad (\lambda^2 - p \text{ irreducible in } R).$$

If one unit then two units outside $\overline{11}$ must exist and we have $[412^1;\ 1]$, $e_1 e_i = e_i$ $(i = 1, 2, 3, 4)$, $e_2 e_1 = e_2$, $e_2^2 = pe_1$, $e_2 e_3 = e_4$, $e_2 e_4 = pe_3$, or its reciprocal.

$[412^1;$ 2 and 3.$]$ These systems have all their units in the group $\overline{11}$. Since $n = 4$ there is a number B in $\overline{11}$ linearly independent of I_1 and A. The fundamental equation of A is

$$\Omega(A) = 0, \qquad [\Omega(\lambda) \equiv \lambda\Omega_1(\lambda) \equiv \lambda(\lambda^2 + p)].$$

Enlarge the domain R by annexing to it $\lambda_1 = -\lambda_2$, the roots of $\lambda^2 + p = 0$, calling the enlarged domain R'. For the domain R' the groups $G_{11}^{(1)}, G_{12}^{(1)}, G_{21}^{(1)}, G_{22}^{(1)}$ replace $\overline{11}$, and are characterized in R' just as $\overline{11}$, $\overline{12}$, $\overline{21}$, $\overline{22}$ were in R; by transformation rational in R' units can be so chosen that each will fall entirely into one or the other of these new groups. There are idempotent numbers $I_1^{(1)}$ and $I_2^{(1)}$ in $\mathfrak{R}(R';\ e_i)$, and in $G_{11}^{(1)}$, $G_{22}^{(1)}$ respectively, such that in $\mathfrak{R}(R;\ e_i)$

$$I_1 = I_1^{(1)} + I_2^{(1)} \qquad \text{and} \qquad A = \lambda_1 I_1^{(1)} + \lambda_2 I_2^{(1)} = \lambda_1 (I_1^{(1)} - I_2^{(1)}).^*$$

If $I_1^{(1)} B I_1^{(1)} \neq 0$, then $I_2^{(1)} B I_2^{(1)} \neq 0$.

For if

$$I_1^{(1)} B I_1^{(1)} = \frac{-pB + ABA + \lambda_1(AB + BA)}{4p} \equiv B_1 \neq 0$$

while

$$I_2^{(1)} B I_2^{(1)} = \frac{-pB + ABA - \lambda_1(AB + BA)}{4p} = 0,$$

we have, on adding and subtracting,

$$\frac{-pB + ABA}{2p} = B_1 = \frac{\lambda_1(AB + BA)}{2p};$$

but since A and B lie in $\mathfrak{R}(R,\ e_i)$, while λ_1 does not lie in R, this equation can only hold if $AB + BA = 0$, and thus if $B_1 = 0$, which is contrary to hypothesis.

Let

$$\bar{B} \equiv B - (I_1^{(1)} B I_1^{(1)} + I_2^{(1)} B I_2^{(1)}).$$

The number $I_1^{(1)} B I_1^{(1)} + I_2^{(1)} B I_2^{(1)}$, which may be zero, lies in $\mathfrak{R}(R,\ e_i)$ and is commutative with A. For

$$I_1^{(1)} B I_1^{(1)} + I_2^{(1)} B I_2^{(1)} = \frac{-pB + ABA}{2p} = \frac{A(AB + BA)}{2p} = \frac{(AB + BA)A}{2p}$$

* TABER, T_2.

Then $\bar{B}$ lies in $\mathfrak{R}(R, e_i)$ and

$$I_1^{(1)}\bar{B}I_1^{(1)} = 0 = I_2^{(1)}\bar{B}I_2^{(1)}.$$

We have $\bar{B} \neq 0$. For if $\bar{B} = 0$ then B is commutative with A; thus

$$B = \frac{-pB + ABA}{2p} = \frac{-pB + A^2B}{2p} = -B,$$

and B is zero, which is contrary to hypothesis. Further $\bar{B}$, I_1, and A are independent numbers in $\mathfrak{R}(R, e_i)$. For if

$$a\bar{B} + bI_1 + cA = 0 \qquad (a, b, c \text{ in } R),$$

we get on using $I_1^{(1)}$ as a prefactor and as a postfactor,

$$b + c\lambda_1 = 0,$$

that is

$$b = c = 0$$

since b and c are, while λ_1 is not, in R. Then I_1, A, $\bar{B}$, $A\bar{B}$, are linearly independent with respect to R, and may be used as units of our system.*

Let

$$B_{12} \equiv I_1^{(1)}\bar{B}I_2^{(1)}, \qquad B_{21} \equiv I_2^{(1)}\bar{B}I_1^{(1)}.$$

Then it can be shown, as in the case of $I_1^{(1)}BI_1'$ above, that if one of B_{12} and B_{21} is zero the other is also. But if $B_{12} = 0 = B_{21}$, then

$$\bar{B} = I_1\bar{B}I_1 = (I_1^{(1)} + I_1^{(2)})\bar{B}(I_1^{(1)} + I_2^{(1)}) = 0,$$

which is impossible. That is neither of B_{12} and B_{21} is zero. Then

$$\bar{B} = I_1\bar{B}I_1 = (I_1^{(1)} + I_2^{(1)})\bar{B}(I_1^{(1)} + I_2^{(1)}) = B_{12} + B_{21},$$

and therefore $A\bar{B} = -\bar{B}A$, since $A = \lambda_1(I_1^{(1)} - I_2^{(1)})$.

The number of units of our system considered as belonging to R' certainly can not exceed the number of units of the system considered as belonging to R. But $I_1^{(1)}$, B_{12}, B_{21}, $I_2^{(1)}$ are linearly independent in $\mathfrak{R}(R'; e_i)$ and may be chosen as units in this domain. Then

$$B_{12}B_{21} = \alpha_{12}I_1^{(1)} \qquad (\alpha_{12} \text{ in } R'),$$

$$B_{21}B_{12} = \alpha_{21}I_2^{(1)} \qquad (\alpha_{21} \text{ in } R'),$$

using the scalar function as developed by H. Taber,† we have

$$\frac{\alpha_{12}}{2} = S(B_{12}B_{21}) = S(B_{21}B_{12}) = \frac{\alpha_{21}}{2}.$$

* See proof of Theorem II.

† Taber, T_1, p. 515.

Therefore

$$\bar{B}^2 = (B_{12} + B_{21})^2 = \alpha_{12}(I_1^{(1)} + I_2^{(1)}) = \alpha_{12} I_1$$

and α_{12} is in R since $\bar{B}$ and I_1 are in $\Re(R, e_i)$. We have two cases $\alpha_{12} = 0$ and $\alpha_{12} \neq 0$.

[412^1; 2]. $e_1 e_i = e_i e_1 = e_i (i = 1, 2, 3, 4)$, $e_2^2 = -pe_1$, $e_2 e_3 = -e_3 e_2 = e_4$ $-e_2 e_4 = e_4 e_2 = pe_3$ ($\lambda^2 + p = 0$ irreducible in R).

[412^1; 3]. $e_1 e_i = e_i e_1 = e_i (i = 1, 2, 3, 4)$, $e_2^2 = -pe_1$, $e_3^2 = -qe_1$, $e_4^2 = -pqe_1$, $e_2 e_3 = -e_3 e_2 = e_4$, $-e_2 e_4 = e_4 e_2 = pe_3$, $e_3 e_4 = -e_4 e_3 = qe_2$ ($\lambda^2 + p = 0$ and $\lambda^2 + q = 0$ irreducible in R).

If R is the domain of real scalars, p and q may be taken as unity; there results the system of Hamiltonian quaternions.

[411^3; 1]. $e_1 e_i = e_i e_1 = e_i$ $(i = 1, 2, 3)$, $e_1 e_4 = e_4$, $e_2^2 = e_3$.

[411^2; 1]. $e_1 e_i = e_i$ $(i = 1, 2, 3, 4)$, $e_2 e_1 = e_2$.

[411^2; 2]. $e_1 e_i = e_i$ $(i = 1, 2, 3, 4)$, $e_2 e_1 = e_2$, $e_2 e_3 = e_4$.

[411^2; 3]. $e_1 e_i = e_i$ $(i = 1, 2, 3)$, $e_i e_1 = e_i$ $(i = 1, 2, 4)$, $e_3 e_4 = e_2$.

[411^2; 4]. $e_1 e_i = e_i$ $(i = 1, 2, 3)$, $e_i e_1 = e_i$ $(i = 1, 2, 4)$.

[411^2; 5]. $e_1 e_i = e_i e_1 = e_i$ $(i = 1, 2, 3)$, $e_1 e_4 = e_4$.

[411^1; 1]. $e_1 e_i = e_i$ $(i = 1, 2, 3, 4)$.

[411^1; 2]. $e_1 e_i = e_i$ $(i = 1, 2, 3)$, $e_i e_1 = e_i$ $(i = 1, 4)$.

Every irreducible hypercomplex number system in less than five units belonging to the domain of rationality R may be transformed by a transformation rational in R either to one of the above systems or to the reciprocal of one of them.

CLARK UNIVERSITY,
May 22, 1905.